Incidence Homology of Finite Projective Spaces

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Abstract

Let $\mathbb{F}$ be the finite field of q elements and let $\mathcal{P}(n,q)$ denote the projective space of dimension n–1 over $\mathbb{F}$. We construct a family $H^n_{k,i}$ of combinatorial homology modules associated to $\mathcal{P}(n,q)$ for a coefficient field F of positive characteristic co-prime to q. As $F\mathrm{GL}(n,q)$-representations these modules are obtained from the permutation action of $\mathrm{GL}(n,q)$ on the set of subspaces of $\mathbb{F}^n$. We prove a branching rule for $H^n_{k,i}$ and use this to determine the homology representations completely. Results include a duality theorem, the characterisation of $H^n_{k,i}$ through the standard irreducibles of $\mathrm{GL}(n,q)$ over F and applications.

Keywords: Incidence homology in partially ordered sets, finite projective spaces, representations of $\mathrm{GL}(n,q)$ in non-defining characteristic, homology representations.

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1 Incidence Homology

Let $\mathcal{P}$ be a finite ranked partially ordered set and let F be a field. In this paper we show that there is a family of homology modules with F as coefficient domain that can be associated to $\mathcal{P}$. This homology contains significant combinatorial and algebraic information when $\mathcal{P}$ is a finite projective space. We determine these modules completely for this case when the coefficient field has positive characteristic co-prime to the characteristic of the space.

The homology modules appear in the following fashion. We assume that the rank function $\mathrm{rk}\colon \mathcal{P} \to \mathbb{N} \cup \{0\}$ is adjusted so that $\min\{\mathrm{rk}(x) : x \in \mathcal{P}\} = 0$. For the integer $k \geq 0$ let $\mathcal{P}_k$ denote the elements of rank k in $\mathcal{P}$ and let $M_k := F\mathcal{P}_k$ be the F-vector space with basis $\mathcal{P}_k$; in particular $M_k = 0$ if $\mathcal{P}_k = \emptyset$.

The partial order on $\mathcal{P}$ now provides a linear *incidence map* $\partial\colon M_k \to M_{k-1}$ defined by $\partial(x) = \sum y$ for $x \in \mathcal{P}_k$ where the sum runs over all $y \in \mathcal{P}_{k-1}$ covered by x. This gives rise to the sequence

$$\mathcal{M}: \quad 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} M_{k-2} \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} M_{k+1} \xleftarrow{\partial} \ldots \xleftarrow{\partial} 0\,. \tag{1}$$

Since $\mathcal{P}$ is finite $\mathcal{M}$ has only finitely many non-zero terms, and in particular, ∂ is nilpotent. Therefore we may choose an integer m such that $\partial^m = 0$ on $\bigoplus_{k\in\mathbb{Z}} M_k$. The options for m depend on both $\mathcal{P}$ and F. If we fix some k and $0 < i < m$ then (1) gives rise to the 'subsequence'

$$\mathcal{M}_{k,i}: \quad 0 \xleftarrow{\partial^*} \ldots \xleftarrow{\partial^*} M_{k-m} \xleftarrow{\partial^*} M_{k-i} \xleftarrow{\partial^*} M_k \xleftarrow{\partial^*} M_{k+m-i} \xleftarrow{\partial^*} M_{k+m} \xleftarrow{\partial^*} \ldots \xleftarrow{\partial^*} 0 \tag{2}$$

in which ∂^* denotes the appropriate power of ∂. Since $\partial^*\partial^* = \partial^m = 0$ it follows that $\mathcal{M}_{k,i}$ is homological. For instance, when $m = 2$ then $\mathcal{M} = \mathcal{M}_{k,i}$ is homological in the usual way. The homology at $M_{k-i} \leftarrow M_k \leftarrow M_{k+m-i}$, denoted by

$$H_{k,i} := (\ker \partial^i \cap M_k) \Big/ \partial^{m-i}(M_{k+m-i}),$$

is the *incidence homology* of $\mathcal{P}$ with coefficient field F for parameters k and i. This construction is canonical in the sense that if G is a group of automorphisms of $\mathcal{P}$ then $H_{k,i}$ is an FG-module. This homology has appeared in various guises before, often under additional restrictions. We mention only some papers [6, 5, 2, 7, 10, 8, 12]. The full details of this construction are explained in Section 3.

In this paper we are interested in the homology of finite projective spaces. Let q be a prime power and $\mathbb{F}$ the field of q elements. For $n \geq 0$ let $\mathcal{P} = \mathcal{P}(n,q)$ be the projective space of dimension $n-1$ over $\mathbb{F}$. As a partially ordered set $\mathcal{P}$ consists of all subspaces of $\mathbb{F}^n$ ordered by containment.

We assume that the coefficient field F has positive characteristic p not dividing q. Here the natural choice for m is the quantum characteristic $m = m(p,q)$ of q in F, see Section 2. We denote $G_n := \mathrm{GL}(n,q)$ and let S_{n-1} be a Singer cycle of G_{n-1}, of order $|S_{n-1}| = q^{n-1} - 1$. For $0 < i < m = m(p,q)$ denote $H^n_{k,i} := H_{k,i}$ and put $H^n_{k,0} = 0 = H^0_{k,m}$. The following branching rule is familiar from James' book [4], it is the key to the incidence homology.

THEOREM 1 (Homology Decomposition). *Let $H^n_{k,i}$ denote the incidence homology of $\mathcal{P}(n,q)$ over a field F of characteristic $p > 0$ not dividing q. Suppose that $0 \leq k \leq n$ with $1 \leq n$ and $0 < i < m = m(p,q)$. Then*

$$H^n_{k,i} \;\cong\; H^{n-1}_{k,i+1} \;\oplus\; H^{n-1}_{k-1,i-1} \;\oplus\; H^{n-2}_{k-1,i}\, S_{n-1}$$

as FG_{n-1}-modules.

We prove a more general version of this theorem in Section 3. It contains an important rule for the index pairs (k,i). We call (k,i) a *middle index for* n provided that $n < 2k + m - i < n + m$. This inequality for the parameters is preserved when passing from one side of the isomorphism to the other. By induction we are able to conclude that $H^n_{k,i}$ vanishes unless (k,i) is a middle index, see Theorem 3 below. The main properties of the homology of projective spaces in co-prime characteristics are summed up in the following two theorems.

THEOREM 2 (Duality). *Assume that (k,i) is a middle index for n and let $j := 2k-n+m-i$ where $m = m(p,q)$. Then*
(i) $H^n_{k,i} \;\cong\; H^n_{n-k,m-i}$ *and*
(ii) $H^n_{k,i} \;\cong\; H^n_{k,j}$
as FG_n-modules.

The incidence homology therefore has a $C_2 \times C_2$ symmetry, we give an example to illustrate this in Section 5 where the theorem is proved. The first part can be interpreted as saying that the duality between k-dimensional and $(n-k)$-dimensional subspaces in projective space remains in place for the homology. At a formal level this can also be understood as a Poincaré duality. The second duality appears to have no immediate geometric interpretation as far as we are aware.

In Section 5 we provide the complete decomposition of the $H^n_{k,i}$ into standard irreducibles of $\mathrm{GL}(n,q)$ over F. To state this result let (k,i) be a middle index for n and define the following parameter intervals

$$\begin{array}{lcll}
T_{k,i} & := & \{t\colon\, k \leq t \leq n-k+i-1\} & \text{if } k \geq \tfrac{1}{2}n \text{ and } i \leq \tfrac{1}{2}(m-n)+k, \\
T_{k,i} & := & \{t\colon\, k \leq t \leq k+m-i-1\} & \text{if } k \geq \tfrac{1}{2}n \text{ and } i > \tfrac{1}{2}(m-n)+k, \\
T_{k,i} & := & \{t\colon\, n-k \leq t \leq n-k+i-1\} & \text{if } k < \tfrac{1}{2}n \text{ and } i \leq \tfrac{1}{2}(m-n)+k, \text{ and} \\
T_{k,i} & := & \{t\colon\, n-k \leq t \leq k+m-i-1\} & \text{if } k < \tfrac{1}{2}n \text{ and } i > \tfrac{1}{2}(m-n)+k.
\end{array}$$

Let λ be a composition of n and let D^λ denote the head of the Specht module S^λ over F. In Sections 4.1 and 5.2 we prove

Theorem 3 (Irreducibles). *Let $0 \le k \le n$ and $0 < i < m = m(p,q)$. Then*

(i) $H^n_{k,i} \ne 0$ if and only if (k,i) is a middle index;

(ii) Let (k,i) be a middle index. Then $H^n_{k,i} = \bigoplus D^{(n-t,t)}$ where the summation runs over all t in $T_{k,i}$.

The first part of this theorem appeared already in [8], the proof here is more direct. We mention that all three theorems above remain true in the limit case '$q = 1$' when the general linear group $\mathrm{GL}(n,\mathbb{F})$ is replaced by the symmetric group $\mathrm{Sym}(n)$ and the 'Singer cycle' of order $q^{n-1} - 1 = 0$ vanishes.

In the case of finite projective spaces the sequence $\mathcal{M}_{k,i}$ in (2) has a remarkable property: For every (k,i) it is *almost exact,* in the sense that it is exact in all but at most one position. A homology arising from $\mathcal{M}_{k,i}$ is non-zero precisely when its index parameters satisfy the middle index condition mentioned before; the details are given in Section 4.

We mention several consequences of this fact. From a standard application of the trace formula it follows that every non-zero homology appears as an alternating sum (in the Burnside ring) of the permutation modules involved in (2), see Corollary 4.3. This in particular provides an explicit character formula for every irreducible $\mathrm{GL}(n,q)$-representation appearing in Theorem 3 in terms of permutation characters, and it also gives an explicit formula for Betti numbers. This is shown in Theorem 4.5. In the same context we mention also the rank modulo p of the incidence matrix of s- versus t-dimensional subspaces of $\mathbb{F}^n$. This has been determined in Frumkin and Yakir [3]. In a forthcoming paper [13] we show that this question has a natural interpretation in terms of the incidence homology of a certain rank-selected poset obtained from the projective space. Other applications [9] concern the multiplicities of irreducible FG-characters when G is an arbitrary subgroup of $\mathrm{GL}(n,\mathbb{F})$ acting on subspaces of $\mathbb{F}^n$. This question was raised by Stanley [14] for ordinary representations over $\mathbb{C}$.

The methods of the paper are elementary, and we use the standard theory of $\mathrm{GL}(n,\mathbb{F})$ representations in cross-characteristics from James' book [4] wherever possible. We thank Alex Zalesskii for helpful comments on an earlier version of this paper.

2 Notation and Prerequisites

Let q be a prime power and $\mathbb{F}$ the field of q elements and let $n \ge 0$ be an integer. For the integer $i \ge 1$ let $[i]_q := 1 + q + \cdots + q^{i-1}$. Then $(i!)_q := [i]_q \cdot [i-1]_q \cdot \ldots \cdot [1]_q$ is the q-factorial of i. If $n \ge k \ge 0$ are integers then the q-binomial coefficient, or Gaussian polynomial, is denoted by

$$\binom{n}{k}_q := \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q} = \frac{[(n)!]_q}{[(n-k)!]_q \cdot [(k)!]_q} \,.$$

This is the number of k-dimensional subspaces of $\mathbb{F}^n$.

Let p be a prime not dividing q and $F := \mathrm{GF}(p)$. Then the integer

$$m = m(p,q) := \min\{\, i > 1 \,:\, [i]_q = 0 \text{ in } F\} \tag{3}$$

is the *(quantum) characteristic* of $\mathbb{F}$ in F. In other words, m is the order of q in $F^\times$ if p does not divide $q-1$ while $m = p$ if p divides $q-1$. In particular, m is the least integer such that $(m!)_q = 0$ in F.

2.1 Permutation action of $\mathrm{GL}(n, \mathbb{F})$ on the set of subspaces of $\mathbb{F}^n$

Let n, q and $\mathbb{F}$ be as above and denote $G_n := \mathrm{GL}(n,q)$. Then G_n acts on $V := \mathbb{F}^n$ (written as row vectors) via the matrix product $g\colon v \mapsto vg$ for v in V and g in G_n. We view G_{n-1} as the subgroup

$$G_{n-1} = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & g' \end{array} \right) \;:\; g' \in \mathrm{GL}(n-1,q) \right\}$$

in G_n and embed $G_{n-2} \subseteq G_{n-1}$ correspondingly. The affine linear group A_{n-1} is the subgroup

$$A_{n-1} := \left\{ \left(\begin{array}{cc} 1 & 0 \\ a & g' \end{array} \right) \;:\; a \in (\mathbb{F}^{n-1})^T,\ g' \in G_{n-1} \right\} .$$

For matrices it is our convention that 1 always stands for a 1×1 submatrix while a 0-entry stands for an appropriate column or row of zeros.

Let $v_1, v_2, \ldots, v_n$ be the standard basis of V. If x is a subspace of V with basis $x_1, \ldots, x_k$ we express the basis vectors as $x_i = \sum_{i=1}^n x_{ij}\, v_j$ in terms of the standard basis. In this way x is represented by the $k \times n$ matrix (x_{ij}) of rank k. It is clear now that G_n acts on the set of subspaces of V via the matrix product

$$g\colon\ x \mapsto xg = (x_{ij})g\ .$$

Another $k\times n$ matrix (x'_{ij}) represents the same x for some other basis $x'_1, \ldots, x'_k$ if and only if (x'_{ij}) is of the form $(x'_{ij}) = h\,(x_{ij})$ for some h in $\mathrm{GL}(k,q)$.

For $0 \le k \le n$ let L^n_k denote the set of all subspaces of dimension k in $V = \mathbb{F}^n$. It is convenient to specify

$$V^{n-1} := \langle v_2, \ldots, v_n\rangle\ , \qquad V^{n-2} := \langle v_3, \ldots, v_n\rangle \quad \text{and}$$

$$L^{n-1}_k := \{\, x \in L^n_k \;:\; x \subseteq V^{n-1} \,\}\,, \qquad L^{n-2}_k := \{\, x \in L^n_k \;:\; x \subseteq V^{n-2} \,\}\,.$$

2.2 Permutation modules of $\mathrm{GL}(n,q)$ on subspaces in cross characteristic

As above let F be a field of characteristic $\mathrm{char}(F) = p > 0$ where we assume that p does not divide $q = |\mathbb{F}|$. If X is an arbitrary set we denote by FX the F-vector space with basis X. If a group G acts on X then FX is the FG-permutation module afforded by G over F. In our situation G will be $G_n = \mathrm{GL}(n,q)$, or a subgroup of it, and X will be some collection of subspaces of $V = \mathbb{F}^n$. This section follows James' book [4] closely. We denote

$$M^n_k := FL^n_k$$

and set $M^n_k = 0$ when $k < 0$ or $k > n$. Similarly let

$$M^{n-1}_k := FL^{n-1}_k \quad \text{and} \quad M^{n-2}_k := FL^{n-2}_k \tag{4}$$

with $M^{n-1}_k = 0 = M^{n-2}_k$ when $L^{n-1}_k = \emptyset$ or $L^{n-2}_k = \emptyset$. It is clear that $M^n_k \supseteq M^{n-1}_k \supseteq M^{n-2}_k$ are modules for FG_n, FG_{n-1} and FG_{n-2} respectively.

Next we assume that $F^\times$ contains an element of order q_0 where q_0 is the prime dividing q. This implies that there is a non-trivial homomorphism

$$\chi\colon (\mathbb{F},+) \to F^\times\ .$$

In FG_n consider the elements

$$E_1 := \frac{1}{q^{n-1}} \sum_{\alpha\in\mathbb{F}^{n-1}} \left(\begin{array}{cc} 1 & 0 \\ \alpha & \mathrm{I}_{n-1} \end{array} \right)$$

and

$$E_2 := \frac{1}{q^{2n-3}} \sum_{\alpha_1 \in \mathbb{F}, \alpha, \beta \in \mathbb{F}^{n-2}} \chi(\alpha_1) \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & 1 & 0 \\ \alpha & \beta & \mathrm{I}_{n-2} \end{pmatrix} .$$

Here I_{n-1} and I_{n-2} are identity matrices, and we emphasize that E_1 and E_2 are elements of FG_n rather than matrix sums. It is easy to show that $E_1E_2 = 0 = E_2E_1$ and $E_iE_i = E_i$ for $i = 1, 2$.

The elements of A_{n-1} commute with E_1 and so, if M is an FG_n-module, then $ME_1 := \langle xE_1 : x \in M\rangle_F$ is an FA_{n-1}-module. Similarly, $ME_2A_{n-1} = \langle xE_2g \ : \ x \in M,\, g \in A_{n-1}\rangle_F$ is an FA_{n-1}-module and this can be written as $ME_2A_{n-1} = ME_2S_{n-1}$ when S_{n-1} denotes a Singer cycle of G_{n-1}. The following theorem of James is the essential branching rule for modules involved in M_k^n.

Theorem 2.1 (Theorems 9.11 and 10.5 in [4]). *Assume that $F^\times$ contains an element of order q_0 where q_0 is the prime dividing $|\mathbb{F}|$. If M is an FG_n-module involved in FL_k^n then*
(i) $M = ME_1 \oplus ME_2S_{n-1}$ *as FA_{n-1}-modules, and*
(ii) $\dim M = \dim(ME_1) + (q^{n-1} - 1)\dim(ME_2)$.

As a corollary we have the following branching rule for M_k^n. The assumption that $\mathrm{char}(F) \neq \mathrm{char}(\mathbb{F})$ is indispensable, it distinguishes the cross-characteristic case from that of defining characteristic.

Theorem 2.2 (Corollary 10.16 in [4]). *If $0 \leq \mathrm{char}(F) \neq \mathrm{char}(\mathbb{F})$ then*

$$M_k^n = M_{k-1}^{n-1} \oplus M_k^{n-1} \oplus M_{k-1}^{n-2} S_{n-1} \tag{5}$$

as FA_{n-1}-modules.

2.3 Incidence maps

The containment relation among subspaces of $V = \mathbb{F}^n$ yields an *incidence map* $\partial : M_k^n \to M_{k-1}^n$ for all k. This map is defined on the basis L_k^n of M_k^n by setting

$$\partial(x) := \sum y$$

where the sum runs over all co-dimension 1 subspaces y of x, for $x \in L_k^n$. If $i \geq 1$ is an integer then

$$\partial^i(x) = c(i) \sum y \tag{6}$$

where the sum runs over all co-dimension i subspaces y of x and where c_i is a coefficient depending only on i. It is easy to see that c_i is the number of chains $y = x_i \subset x_{i-1} \subset \cdots \subset x_1 \subset x$ of length i, and hence

$$c(i) = [(i)!]_q = [i]_q \cdot [i-1]_q \cdots [1]_q .$$

Clearly ∂^i is an FG_n-homomorphism, and in particular, the idempotents E_i defined above commute with ∂^i. The following is useful for computations. If x, y are subspaces of V let $x \cdot y$ denote the subspace spanned by x and y. In particular, we write $v_1 \cdot y$ for the subspace $\langle v_1, y\rangle$. This can be extended linearly to an associative and commutative product on $M^n := \bigoplus M_k^n$.

Let $0 \leq k \leq n$ and $f \in M_k^n$. Then there are unique elements $f_1 \in M_{k-1}^{n-1}$ and $\ell \in M_k^{n-1}$ so that

$$f = v_1 \cdot f_1 + \ell .$$

This is the *standard decomposition* of f. We collect some useful facts.

Lemma 2.3. *(i) Let $f_1 \in M_{k-1}^{n-1}$. Then $(v_1 \cdot f_1)E_1 = v_1 \cdot f_1$ and $(v_1 \cdot f_1)E_2 = 0$.*
(ii) Let $x = (a|x') \in L_k^n$ with $a \in (\mathbb{F}^k)^T$ and $x' \in L_k^{n-1}$ be a space not containing v_1. Then $xE_1 = q^{-k} \sum (b|x')$ where the sum runs over all $b \in (\mathbb{F}^k)^T$. Furthermore, $xE_2 = 0$ unless $x' = v_2 \cdot x''$ with $x'' \in L_{k-1}^{n-2}$.

Proof: (i) The first part is a simple calculation, and for the second part we have $(v_1 \cdot f_1)E_1 = v_1 \cdot f_1$ so that $(v_1 \cdot f_1)E_2 = (v_1 \cdot f_1)E_1E_2 = 0$. (ii) This is also a direct calculation, alternatively see Theorem 10.2 in [4] for a more general case. □

LEMMA 2.4. *Let* $f_1 \in M^{n-1}_{k-1}$ *and* $i \geq 1$. *Then* $\partial^i(v_1 \cdot f_1) = v_1 \cdot \partial^i(f_1) + q^{k-i}[i]_q\partial^{i-1}(f_1E_1)$.

Proof: By linearity we can assume that $f_1 = x' \in L^{n-1}_{k-1}$ and so $\partial^i(v_1 \cdot x')$ represents the sum in M^n_{k-i} of all $(k-i)$-dimensional subspaces of $v_1 \cdot x'$ with coefficient $[(i)!]_q$, see (6). On the right hand side of the equation we have subspaces y of $v_1 \cdot x'$ of dimension $k-i$, and y has the correct coefficient $[(i)!]_q$ if it contains v_1. Otherwise y is a summand in the second term on the right hand side. In this case its coefficient is $q^{k-i}[i]_q[(i-1)!]_q \cdot q^{-(k-i)}$ see Lemma 2.3(ii). Now $q^{k-i}[i]_q[(i-1)!]_q \cdot q^{-(k-i)} = [(i)!]_q$ as required. □

3 Partially Ordered Sets

Let $(\mathcal{P}, \leq)$ be a finite ranked poset with rank function $\mathrm{rk}\colon \mathcal{P} \to \mathbb{N}\cup\{0\}$, and assume that $\min\{\mathrm{rk}(x) : x \in \mathcal{P}\} = 0$. Let G be the automorphism group of $\mathcal{P}$ and let F be a coefficient ring with 1. We describe a construction that associates to this data a family of homology modules which arise from the FG-action on certain permutation modules obtained from $\mathcal{P}$.

For $k \in \mathbb{Z}$ denote the set of all $x \in \mathcal{P}$ with $\mathrm{rk}(x) = k$ by $\mathcal{P}_k$ and let $M_k := F\mathcal{P}_k$ be the F-vector space with basis $\mathcal{P}_k$. In particular $M_k = 0$ when $k < 0$ or $k > \max\{\mathrm{rk}(x) : x \in \mathcal{P}\}$. Consider the linear *incidence map* $\partial\colon M_k \to M_{k-1}$ which is defined for $x \in \mathcal{P}_k$ by

$$\partial(x) = \sum y$$

where the sum runs over all $y \in \mathcal{P}_{k-1}$ with $y \leq x$. Then ∂ is an FG-homomorphism on $M = \bigoplus_{k\in\mathbb{Z}} M_k$ and gives rise to the FG-sequence

$$\mathcal{M}: \quad 0 \xleftarrow{\partial} M_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} M_{k-2} \xleftarrow{\partial} M_{k-1} \xleftarrow{\partial} M_k \xleftarrow{\partial} M_{k+1} \xleftarrow{\partial} \ldots \xleftarrow{\partial} 0\,. \tag{7}$$

From this sequence we obtain homological FG-sequences as follows. Note that $M_k \neq 0$ for only finitely many k, as $\mathcal{P}$ is finite, and hence ∂ is nilpotent. Hence let $m > 1$ be an integer for which $\partial^m = 0$ as a map on M. Fixing some $0 \leq k$ and $0 < i < m$ now consider the sequence

$$\mathcal{M}_{k,i}: \quad \ldots \xleftarrow{\partial^*} M_{k-m} \xleftarrow{\partial^*} M_{k-i} \xleftarrow{\partial^*} M_k \xleftarrow{\partial^*} M_{k+m-i} \xleftarrow{\partial^*} M_{k+m} \xleftarrow{\partial^*} \ldots \tag{8}$$

in which $\partial^* = \partial^i$ or ∂^{m-i} stands for the appropriate power of ∂. Then $\mathcal{M}_{k,i}$ is homological since $\partial^*\partial^* = \partial^m = 0$. We denote the homology at $M_{k-i} \leftarrow M_k \leftarrow M_{k+m-i}$ by

$$H_{k,i} := (\ker \partial^i \cap M_k) \Big/ \partial^{m-i}(M_{k+m-i})\ .$$

We call $H_{k,i}$ the *incidence homology* of $\mathcal{P}$ over F for parameters k and $0 < i < m$. For example, when $m = 2$ then $\mathcal{M} = \mathcal{M}_{k,1}$ for any $k \geq 0$ is a homological sequence in the usual sense. This definition of the homology depends on m, and the choices for m depend on both F and $\mathcal{P}$. For instance, one may take m to be minimal with the property $\partial^m = 0$, but this is not the only case to consider.

The homology modules are related to each other. Denote $I_{k,i} := \partial^{m-i}(M_{k+m-i})$ and $K_{k,i} := (\ker \partial^i \cap M_k)$. Then ∂ induces a linear map

$$\partial : H_{k,i} \to H_{k-1,i-1} \quad \text{by} \quad \partial(x + I_{k,i}) = \partial(x) + I_{k-1,i-1}\ . \tag{9}$$

In a similar way the identity map $\mathrm{inc}\colon M_k \to M_k$ induces a map $\mathrm{inc}\colon H_{k,i} \to H_{k,i+1}$. If the $H_{k,i}$ are arranged as a grid with rows indexed by $0 < i < m$ and columns indexed by k then ∂ and inc

connect the modules in the array

$$
\begin{array}{ccc}
H_{k,i} & & H_{k+1,i} \\
& \nwarrow^{\partial} & \downarrow^{\mathrm{inc}} \\
& & H_{k+1,i+1}
\end{array}
\tag{10}
$$

of maps which carries essential information. (This feature is particular, it does not exist in ordinary homology.) It is used to determine completely the homologies of finite projective spaces in Section 5.

We emphasize that this construction is entirely general, it applies to an arbitrary finite ranked poset for any arbitrary coefficient ring with 1.

3.1 Branching Rules in Projective Space

We now consider this homology for finite projective spaces. Let q be a prime power, $\mathbb{F}$ the field of q elements and $n \geq 0$. Then the projective space $\mathcal{P} = \mathcal{P}(n, q)$ is the set of all subspaces of $\mathbb{F}^n$ ordered by inclusion, with rank function given by dimension. Let $G_n := \mathrm{GL}(n, q)$. As coefficients we choose a field F of characteristic $p > 0$ not dividing q and let

$$m = m(p, q)$$

be the characteristic of $\mathbb{F}$ in F discussed at the beginning of Section 2.

We use the same notation for modules and incidence maps as above and as in Sections 2.2 and 2.3. It is clear from (6) that $\partial^m = 0$ on $M = \bigoplus_{k\in\mathbb{Z}} M_k$ independently of n. In particular, if $0 \leq k$ and $0 < i < m$ are given then we obtain the homological FG-sequence (or FG-chain complex)

$$\mathcal{M}^n_{k,i}: \quad \ldots \xleftarrow{\partial^*} M^n_{k-m} \xleftarrow{\partial^*} M^n_{k-i} \xleftarrow{\partial^*} M^n_k \xleftarrow{\partial^*} M_{k+m-i} \xleftarrow{\partial^*} M_{k+m} \xleftarrow{\partial^*} \ldots \quad . \tag{11}$$

The homology for parameters k and $0 < i < m$ is denoted by

$$H^n_{k,i} := K^n_{k,i} \Big/ I^n_{k,i}.$$

It is convenient to set $K^n_{k,0} = 0$, $K^n_{k,m} = M^n_k$ and $I^n_{k,0} = 0$, $I^n_{k,m} = M^n_k$. In particular, $H^n_{k,0} = 0 = H^n_{k,m}$ for all k and n.

It is appropriate to discuss homology at the level of FG-sequences, rather than individual modules. (We prefer the term 'FG-sequence' instead of 'FG-chain complex' as it is simpler and does not interfere with the notion of a combinatorial complex.)

Recall that if $\mathcal{A}$: ... $\xleftarrow{\alpha} A_{k-1} \xleftarrow{\alpha} A_k \xleftarrow{\alpha}$ ... and $\mathcal{B}$: ... $\xleftarrow{\beta} B_{k-1} \xleftarrow{\beta} B_k \xleftarrow{\beta}$ ... are finite FG-sequences then $\mathcal{A} \cong \mathcal{B}$ are isomorphic if and only if there is an FG-isomorphism $\psi\colon A_j \to B_j$ for all j such that $\psi\alpha = \beta\psi$. (Assume that the first non-zero modules are A_0 and B_0 respectively.) Similarly, if $\mathcal{C}$: ... $\xleftarrow{\gamma} C_{k-1} \xleftarrow{\gamma} C_k \xleftarrow{\gamma}$ ... is an FH-sequence for some subgroup $H \subseteq G$ and if $A_k = C_k S$ is obtained by induction from H to G for each k for some $S \subseteq G$ then $\mathcal{A} = \mathcal{C}S$ is the FG-sequence induced from $\mathcal{C}$ with maps induced from γ.

Let S_{n-1} denote a Singer cycle in $G_{n-1} = \mathrm{GL}(n-1, q)$, of order $|S_{n-1}| = q^{n-1} - 1$, and let $M^{n-2}_{k-1} S_{n-1}$ be the module induced from G_{n-2} to G_{n-1} via Harish-Chandra induction. Correspondingly let $\mathcal{M}^{n-2}_{k-1,i} S_{n-1}$ be the FG_{n-1}-sequence induced from $\mathcal{M}^{n-2}_{k-1,i}$.

Theorem 3.1 (Branching Rule). *Let $\mathbb{F}$ be the field of q elements and let F be a field of characteristic $p > 0$ not dividing q. Suppose that $0 \leq k \leq n$ with $1 \leq n$ and $0 < i < m = m(p,q)$ are integers. Then*

$$\mathcal{M}^n_{k,i} \;\cong\; \mathcal{M}^{n-1}_{k,i+1} \;\oplus\; \mathcal{M}^{n-1}_{k-1,i-1} \;\oplus\; \mathcal{M}^{n-2}_{k-1,i}\, S_{n-1}$$

as FG_{n-1}-sequences.

This result is Theorem 2.2 at the level of FG_{n-1}-sequences; the proof is given in the next section. The following diagram may clarify the the situation. We show that there is an isomorphism ϑ of FG_{n-1}-sequences so that all maps commute.

$$\begin{tikzcd}
M^n_{k-i+m} \arrow[r, "\vartheta_{k-i+m}"] \arrow[d, "\partial^{m-i}"] & M^{n-1}_{k-i+m} \arrow[drr, "\partial^{m-i-1}"] & \oplus & M^{n-1}_{k-i+m-1} \arrow[dll, "\partial^{m-i+1}"'] & \oplus & M^{n-1}_{k-i+m-1}S_{n-1} \arrow[d, "\partial^{m-i}"] \\
M^n_k \arrow[r, "\vartheta_k"] \arrow[d, "\partial^{i}"] & M^{n-1}_k \arrow[drr, "\partial^{i-1}"] & \oplus & M^{n-1}_{k-1} \arrow[dll, "\partial^{i+1}"'] & \oplus & M^{n-2}_{k-1}S_{n-1} \arrow[d, "\partial^{i}"] \\
M^n_{k-i} \arrow[r, "\vartheta_{k-i}"] & M^{n-1}_{k-i} & \oplus & M^{n-1}_{k-i-1} & \oplus & M^{n-2}_{k-i-1}S_{n-1}
\end{tikzcd}$$

Figure 1: Branching Rule

Since $\mathcal{M}^n_{k,i}$ is homological we have corresponding FG-sequences for images and kernels, and this gives the sequence

$$\mathcal{H}^n_{k,i}: \quad \ldots \xleftarrow{\partial^*} H^n_{k-i,m-i} \xleftarrow{\partial^*} H^n_{k,i} \xleftarrow{\partial^*} H^n_{k+m-i,m-i} \xleftarrow{\partial^*} \ldots \quad . \tag{12}$$

Using Theorem 3.1 and standard results from homological algebra we have the following corollary:

Theorem 3.2 (Homology Decomposition). *Let $\mathbb{F}$ be the field of q elements and let F be a field of characteristic $p > 0$ not dividing q. Suppose that $0 \leq k \leq n$ and $0 < i < m = m(p,q)$ are integers. Then*

$$\mathcal{H}^n_{k,i} \;\cong\; \mathcal{H}^{n-1}_{k,i+1} \;\oplus\; \mathcal{H}^{n-1}_{k-1,i-1} \;\oplus\; \mathcal{H}^{n-2}_{k-1,i}\, S_{n-1}$$

as FG_{n-1}-sequences.

This in turn completes the proof of Theorem 1.

3.2 Proof of the Branching Rule

The proof requires two simple observations. First consider the idempotent E_1 introduced in Section 2.2. Observe that ∂ restricts to a map $\partial : M^n_k E_1 \to M^n_k E_1$ and hence we may consider the FG_{n-1}-sequence $\mathcal{M}^n_{k,i}E_1$. We have

Lemma 3.3. *Suppose that $0 \leq k \leq n$ and $0 < i < m$ are integers. Then*

$$\mathcal{M}^n_{k,i}E_1 \;\cong\; \mathcal{M}^{n-1}_{k,i+1} \;\oplus\; \mathcal{M}^{n-1}_{k-1,i-1}$$

as FG_{n-1}-sequences.

Proof: If $0 \le j \le n$ is an integer and f belongs to $M_j^n E_1$ consider its standard decomposition $f = v_1 \cdot f_1 + \ell$ as discussed in Section 2.3. Thus f_1 belongs to M_{j-1}^{n-1} and none of the spaces appearing in ℓ contains v_1. Hence ℓ is of the shape

$$\ell = \sum \ell_{b,x'} \left(b \,\middle|\, x'\right)$$

where the sum runs over all $b \in (\mathbb{F}^k)^\perp$ and $x' \in L_k^{n-1}$, see Lemma 2.3. Hence let

$$\ell^* := \sum \ell_{b,x'} \; x' \; \in M_k^{n-1} \, .$$

(Explicitly, ℓ^* is obtained by removing the first column of all matrices appearing in ℓ.) We define the map $\psi_j \colon M_j^n E_1 \to M_j^{n-1} \oplus M_{j-1}^{n-1}$ for $j = k+m-i,\ k,\ k-i, \ldots$ by

$$\begin{aligned}
\psi_{k-i+m}(f) &= \ell^* + q^k[m-i]_q f_1 + \partial(\ell^*) \\
\psi_k(f) &= \ell^* + q^{k-i}[i]_q f_1 + \partial(\ell^*) \\
\psi_{k-i}(f) &= \ell^* + q^{k-m}[m-i]_q f_1 + \partial(\ell^*) \, , \quad \text{etc.} \, ,
\end{aligned}$$

see again Figure 1. It is clear that ψ is an injective FG_{n-1}-homomorphism. By Theorems 2.1 and 2.2 we have $\dim M_k^n E_1 = \dim M_k^{n-1} + \dim M_{k-1}^{n-1}$ and hence ψ is an isomorphism. To prove that ψ is a homomorphism of sequences it suffices to examine the case $j = k$, the other indices are the same. To show that $(\partial^{i+1} \oplus \partial^{i-1})(\psi_k(f)) = \psi_{k-i}(\partial^i(f))$ we obtain

$$\begin{aligned}
(\partial^{i+1} \oplus \partial^{i-1})(\psi_k(f)) &= \partial^{i-1}\big(q^{k-i}[i]_q f_1 + \partial(\ell^*)\big) \; + \; \partial^{i+1}(\ell^*) \\
&= q^{k-i}[i]_q \partial^{i-1}(f_1) + \partial^i(\ell^*) \; + \; \partial^{i+1}(\ell^*) \, .
\end{aligned}$$

For the standard decomposition of $\partial^i(f)$ we have $\partial^i(v_1 \cdot f_1 + \ell) = v_1 \cdot \partial^i(f_1) + q^{k-i}[i]_q \partial^{i-1}(f_1 E_1) + \partial^i(\ell)$ by Lemma 2.4 and therefore

$$\begin{aligned}
\psi_{k-i}\big(\partial^i(v_1 \cdot f_1 + \ell)\big) &= q^{k-i}[i]_q \partial^{i-1}(f_1) + \partial^i(\ell^*) + \big(q^{k-m}[m-i]_q + q^{k-i}[i]_q\big)\partial^i(f_1) \\
&\quad + \partial^{i+1}(\ell^*).
\end{aligned}$$

The result follows since $q^{k-m}[m-i]_q + q^{k-i}[i]_q = q^{k-m}(1 + q + \ldots + q^{m-1}) = 0$ in F. □

Next we consider the idempotent E_2; in particular, assume that $F^\times$ contains an element of order q_0 where q_0 is the prime dividing q. Here we have the FG_{n-2}-sequence $\mathcal{M}_{k,i}^n E_2$, we claim that there is the following isomorphism:

Lemma 3.4. *Suppose that $0 \le k \le n$ and $0 < i < m$ are integers. Then*

$$\mathcal{M}_{k,i}^n E_2 \;\cong\; \mathcal{M}_{k-1,i}^{n-2}$$

as FG_{n-2}-sequences.

Proof: Let $0 \le j \le n$. Then the elements $(v_2 \cdot x'')E_2$ with $x'' \in L_{j-1}^{n-2}$ form a basis of $M_j^n E_2$, see Lemma 2.4. For $j = k+m-i,\ k,\ k-i, \ldots$ define the map $\varphi_j \colon M_j^n E_2 \to M_{j-1}^{n-2}$ by

$$\varphi_j((v_2 \cdot x'')E_2) = x'' \;\in\; M_{k-1}^{n-2} \, .$$

(Explicitly, φ removes the first two columns in all matrices appearing in $f \in M_{k,i}^n E_2$.) It is immediate that φ is an FG_{n-2}-isomorphism, and similarly that φ commutes with ∂^i and ∂^{m-i} as appropriate. □

Proof of Theorem 3.1: First assume that $F^\times$ contains an element of order q_0. Let S_{n-1} be a Singer cycle of G_{n-1}. By the earlier comment on induced sequences and Lemma 3.4 we have an isomorphism

$\mathcal{M}^n_{k,i}E_2S_{n-1} \cong \mathcal{M}^{n-2}_{k-1,i}S_{n-1}$ of FG_{n-1}-sequences which we also denote by φ. Using Lemma 3.3 we therefore have an isomorphism

$$\vartheta := (\psi, \varphi)\colon\ \mathcal{M}^n_{k,i}E_1 \oplus \mathcal{M}^n_{k,i}E_2S_{n-1} \cong \mathcal{M}^{n-1}_{k,i+1} \oplus \mathcal{M}^{n-1}_{k-1,i-1} \oplus \mathcal{M}^{n-2}_{k-1,i}S_{n-1}$$

of FG_{n-1}-sequences. Finally $\mathcal{M}^n_{k,i} = \mathcal{M}^n_{k,i}E_1 \oplus \mathcal{M}^n_{k,i}E_2S_{n-1}$ by Theorem 2.1 since ∂^* commutes with E_1 and E_2. This proves the theorem when $F^\times$ contains an element of order q_0. In the remaining case, extend F to a field $\bar{F} \supset F$ which contains an element of order q_0 and apply the result to this larger field. Now notice that the map (ψ, φ) restricts back to a map over F. □

4 Homology Modules $H^n_{k,i}$

We begin to analyze the incidence homology for finite projective spaces in detail. As before $\mathbb{F} = \mathrm{GF}(q)$ and F is a field of characteristic $p > 0$ not dividing q. Denote $G_n = \mathrm{GL}(n, q)$ and let $m = m(p, q)$ be the characteristic of q in F.

Let $n \geq 0$. For any $k \leq n$ and i with $0 < i < m$ we have the homological sequence

$$\mathcal{M}^n_{k,i}:\quad 0 \stackrel{\partial^*}{\leftarrow} \ldots \stackrel{\partial^*}{\leftarrow} M^n_{k-m} \stackrel{\partial^*}{\leftarrow} M^n_{k-i} \stackrel{\partial^*}{\leftarrow} M^n_k \stackrel{\partial^*}{\leftarrow} M^n_{k+m-i} \stackrel{\partial^*}{\leftarrow} \ldots \stackrel{\partial^*}{\leftarrow} 0 \qquad (13)$$

in which ∂^* is the appropriate power of ∂. Suppose that $n \geq m$. Then for any choice of $a < b$ in $\{0, .., m-1\}$ the sequence $\mathcal{M}^n_{b,b-a}$ contains the term $M^n_a \stackrel{\partial^*}{\leftarrow} M^n_b$, and all $\mathcal{M}^n_{k,i}$ are of this form. Hence there are $\binom{m}{2}$ distinct sequences if $n \geq m$. For instance, the index pairs (k, i), $(k-i, m-i)$, and $(k \pm m, i)$ etc. all define the same sequence. We write $(k, i) \sim (k', i')$ if $\mathcal{M}^n_{k,i} = \mathcal{M}^n_{k',i'}$.

4.1 Almost exact sequences and Brauer characters

If $\mathcal{A}$ is a homological sequence then its homology is *concentrated in a single position*, or $\mathcal{A}$ is *almost exact,* if all but at most one of the homology modules in $\mathcal{A}$ vanish. We show that $\mathcal{M}^n_{k,i}$ has this property for all (k, i). One direction of the following is Theorem 3.1 in the paper [8] with Valery Mnukhin.

Theorem 4.1. *Let $0 \leq k \leq n$ and $0 < i < m$. Then $H^n_{k,i} \neq 0$ if and only if $n < 2k+m-i < n+m$.*

We call (k, i) a *middle index* for n if (a) $0 \leq k \leq n$, (b) $0 < i < m$ and (c) $n < 2k+m-i < n+m$. If (k, i) is a middle index then $M^n_{k-i} \leftarrow M^n_k$ is the *middle index* of the (unique) sequence in which the two modules appear. It is easy to check that for any k', i' the sequence $\mathcal{M}^n_{k',i'}$ contains at most one middle index. But it may contain none. When $m = 2$, for instance, then there is a unique middle index $(k, 1)$ for n even, and $k = \frac{n}{2}$ in this case, while there is no middle index when n is odd.

Proof of Theorem 4.1: Assume that $0 < i < m$ and $0 \leq k \leq n$. Writing out the inequalities observe that (k, i) is a middle index for n if and only if $(k, i+1)$, $(k-1, i-1)$ and $(k-1, i)$ are middle indices for $n-1$, $n-1$ and $n-2$, respectively, unless $i \in \{1, m-1\}$ or $k = 0$. The result now follows from Theorem 1 and induction on n. □

Corollary 4.2. *For all (k, i) with $0 < i < m$ the sequence $\mathcal{M}^n_{k,i}$ is almost exact. Furthermore, $\mathcal{M}^n_{k,i}$ is exact if and only $(k, i) \sim (k', i')$ implies that (k', i') is not a middle index for n.*

This corollary implies that the incidence homology of $\mathcal{P}(n, q)$ lies in the Burnside ring of $\mathrm{GL}(n, q)$ over F. For each (k, i) the sequence $\mathcal{M}^n_{k,i}$ is homological and therefore the Hopf-Lefschetz trace formula (see for instance Theorem 22.1, Chapter 2 in Munkres [11]) says that

$$\bigoplus_{t\in\mathbb{Z}} \left(H^n_{k+tm,i} - H^n_{k-i+tm,m-i} \right) = \bigoplus_{t\in\mathbb{Z}} \left(M^n_{k+tm} - M^n_{k-i+tm} \right) \qquad (14)$$

If (k, i) is the middle index of $\mathcal{M}^n_{k,i}$ then $H^n_{k,i}$ is the only non-trivial homology in (14) and hence Corollary 4.2 gives

Corollary 4.3 (Trace Formula). *Let* (k,i) *be a middle index. Then*

$$H^n_{k,i} = \bigoplus_{t\in\mathbb{Z}} M^n_{k+tm} - \bigoplus_{t\in\mathbb{Z}} M^n_{k-i+tm} \tag{15}$$

as FG_n *-modules in the Burnside ring.*

Considering characters, if H is any FG_n-module, let $\chi(g,\,H)$ denote the Brauer character of G_n on H. In particular, the permutation character

$$\chi(g,\,M^n_k) = \pi_k(g)$$

is the number of k-dimensional spaces of V that are stabilized by $g\in G_n$. Hence Corollary 4.3 yields

Theorem 4.4. *Let* (k,i) *be a middle index for* n*. Then the Brauer character of* $\mathrm{GL}(n,q)$ *on* $H^n_{k,i}$ *is*

$$\chi(g,\,H^n_{k,i}) \;=\; \sum_{t\in\mathbb{Z}} \;\pi_{k+tm}(g) - \pi_{k-i+tm}(g)\,.$$

This key fact is already mentioned in [8], it also provides the dimensions of the homology modules in the next section.

4.2 Betti numbers

By Theorem 4.1 we have $H^n_{k,i}\neq 0$ if and only if (k,i) is a middle index, and in that case let

$$\beta^n_{k,i} := \dim H^n_{k,i}$$

be the Betti number of $H^n_{k,i}$. As before it will be useful to set $\beta^n_{k,0}=0=\beta^n_{k,m}$.

Theorem 4.5. *Let* (k,i) *be a middle index for* n*. Then*

(i) $\beta^n_{k,i} = \beta^{n-1}_{k,i+1} \;+\; \beta^{n-1}_{k-1,i-1} \;+\; \beta^{n-2}_{k-1,i}(q^{n-1}-1)\,;$

(ii) $\beta^n_{k,i} = \sum_{t\in\mathbb{Z}} \binom{n}{k+tm}_q - \binom{n}{k-i+tm}_q$*. In particular,* $\beta^n_{k,i}$ *is the Euler characteristic of* $\mathcal{M}^n_{k,i}$*;*

(iii) (Duality:) For all $0\le\ell\le n$ *and* $0<j<m$ *we have* $\beta^n_{\ell,j}=\beta^n_{n-\ell,m-j}$*.*

(iv) Let $\epsilon=2$ *when* n *is odd and* $\epsilon=1$ *otherwise. Then as a polynomial in* q *we have*

$$\beta^n_{k,i} = (q^{n-1}-1)(q^{n-3}-1)(q^{n-5}-1)\cdots(q^{\epsilon}-1)+f(q)$$

where $f(q)$ *is a polynomial of degree* $<(n-1)+(n-3)+(n-5)+\cdots+\epsilon$*.*

Remark: We say that the index pair $(n-\ell,m-j)$ is *dual* to (ℓ,j). The equation $\beta^n_{\ell,j}=\beta^n_{n-\ell,m-j}$ in (iii) extends to an isomorphism $H^n_{\ell,j}\cong H^n_{n-\ell,m-j}$. This is shown in Section 5 which contains many additional inequalities for Betti numbers.

Proof: The first part (i) follows from Theorem 1 and (ii) follows from (15). To show (iii) first notice that (ℓ,j) is a middle index if and only if the same is true about its dual. Hence we assume that (ℓ,j) is a middle index and proceed by induction. The equality $\beta^n_{\ell,j}=\beta^n_{n-\ell,m-j}$ is easily checked for $n\le 2$. Using (i) we have

$$\beta^n_{\ell,j} = \beta^{n-1}_{\ell,j+1}+\beta^{n-1}_{\ell-1,j-1}+\beta^{n-2}_{\ell-1,j}(q^{n-1}-1)$$

and

$$\beta^n_{n-\ell,m-j} = \beta^{n-1}_{n-1-\ell,m-j-1}+\beta^{n-1}_{n-1-\ell+1,m-j+1}+\beta^{n-2}_{n-2-\ell+1,m-j}(q^{n-1}-1)\,.$$

Observe that $(\ell-1, j)$ is dual to $(n-2-\ell+1, m-j)$ relative to $n-2$ and the other two pairs are dual to each other relative to $n-1$. The result now follows by induction. The last part follows by induction on n from the first part. □

For small $m(p,q)$ one can analyze the Betti numbers a little further.

COROLLARY 4.6. *(a) Suppose that $m(p,q)=2$. If n is odd then $\mathcal{M}^n_{k,1}$ is exact. If $n=2k$ is even then $\beta^n := \beta^n_{k,1}$ depends on n only and*

$$\beta^n = (q^{n-1}-1)(q^{n-3}-1)(q^{n-5}-1)\cdots(q-1).$$

(b) Suppose that $m(p,q)=3$. For given n either $\mathcal{M}^n_{k,i}$ is exact or (k,i) is one of two middle indices $(k,1)\neq(k',2)$ when $\beta^n := \beta^n_{k,1} = \beta^n_{k',2}$ depends on n only. Furthermore,

$$\beta^n = \beta^{n-1} + \beta^{n-2}(q^{n-1}-1)$$

with initial values $\beta^0 = \beta^1 = 1$.

Proof: We leave this to the reader. □

5 Module Structure and Duality

We determine the incidence homology up to isomorphism, in terms of the standard $\mathrm{GL}(n,q)$-irreducibles over F. In addition we show that there are dualities between homology modules for certain index pairs.

5.1 Composition Factors

If (k,i) is a middle index for n then $n < 2k+m-i < n+m$. We say that (k,i) is a *maximal middle index* if $2k-i = n-1$. The following is checked easily.

LEMMA 5.1. *Let $n \geq 2$ and let (k,i) be a maximal middle index for n. Then*
(i) $(k-1,i)$ *is a maximal middle index for $n-2$ unless $k=0$;*
(ii) $(k,i+1)$ *is a maximal middle index for $n-1$ unless $i+1=m$;*
(iii) $(k-1,i-1)$ *is a maximal middle index for $n-1$ unless $i-1=0$ or $k=0$.*

If $\lambda = (\lambda_1,\lambda_2)$ is a composition of n denote the Specht module for λ by S^λ and let $D^\lambda := S^\lambda / S^\lambda \cap (S^\lambda)^\perp$ be its head. Then $D^\lambda = D^{(\lambda_2,\lambda_1)}$ is the usual standard irreducible FG_n-modules indexed by λ. If $\mu = (\mu_1,\mu_2)$ is another composition of n then λ majorizes μ, denoted $\lambda \geq \mu$, if a largest part of λ is bigger or equal to a largest part of μ. We write $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$, see James' book [4].

PROPOSITION 5.2. *(i) Let (k,i) be a middle index. Then all composition factors of $H^n_{k,i}$ have the form D^λ with $\lambda \geq (n-k,k)$. Furthermore, $D^{(n-k,k)}$ has multiplicity 1 in $H^n_{k,i}$.*
(ii) If (k,i) is a maximal middle index for n then $H^n_{k,i} \cong D^{(n-k,k)}$.

Proof: (i) By Corollary 16.3 in [4] the composition factors of M^n_k are of the form D^λ with $\lambda \geq (n-k,k)$ and hence the same is true for $H^n_{k,i}$. Furthermore, $D^{(n-k,k)}$ has multiplicity 1 in M^n_k. The remainder is easy to verify when $n \leq 2$ and when $k=0$. In the latter case we have $H^n_{k,i} = F = D^{(n,0)}$. If $n \geq 2$ and if (k,i) is a middle index of n then $(k-1,i)$ is a middle index of $n-2$. By induction and the homology decomposition we may assume that $D^{(n-1-k,k-1)}$ is a composition factor of $H^{n-2}_{k-1,i}$.

Let D^{λ_t}, $D^{\lambda_{t-1}}, \ldots$ D^{λ_1} be the composition factors of $H^n_{k,i}$, ordered so that $\lambda_t > \lambda_{t-1} > \ldots > \lambda_1 \geq (n-k,k)$. For $\lambda = (a,b)$ with $a, b > 0$ let λ'' be the composition $(a-1, b-1)$ of $n-2$. It follows from Theorem 16.9 of [4] that the composition factors of $H^{n-2}_{k-1,i} = H^n_{k,i} E_2$ are of the shape $D^{\lambda''_j}$ with $\lambda''_j \geq (n-k,k)'' = (n-1-k, k-1)$. As $D^{(n-1-k,k-1)}$ is a composition factor of $H^{n-2}_{k-1,i}$ we have $\lambda_1 = (n-k,k)$. By Corollary 16.3 the multiplicity of $D^{(n-k,k)}$ in $H^n_{k,i}$ is 1.

(ii) This is easy to verify when $n \leq 2$ or when $k = 0$. So suppose that $n \geq 2$ and $k \neq 0$. Using Lemma 5.1 we may apply induction to the terms on the right hand side of the decomposition

$$H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1} \oplus H^{n-2}_{k-1,i} S_{n-1}.$$

By the first part of the theorem we have $H^{n-2}_{k-1,i} = D^{(n-k,k)''}$ since $k > 0$. Furthermore, we have $H^{n-1}_{k,i+1} = D^{(n-k,k)^a}$ (unless $H^{n-1}_{k,i+1} = 0$ when $i = m-1$) and $H^{n-1}_{k-1,i-1} = D^{(n-k,k)^b}$ (unless $H^{n-1}_{k-1,i-1} = 0$ when $i = 1$) where $(n-k,k)^a := (n-1-k,k)$ and $(n-k,k)^b := (n-k,k-1)$ are the corresponding compositions of $n-1$. By the first part we have that $D^{(n-k,k)}$ has multiplicity one in $H^n_{k,i}$. Suppose that also D^{λ} with $\lambda > (n-k,k)$ appeared as a composition factor in $H^n_{k,i}$. Now use Theorem 16.9 of [4] (or direct computation) to show that at least one of the modules $H^{n-1}_{k,i+1}$, $H^{n-1}_{k-1,i-1}$, $H^{n-2}_{k-1,i}$ has more than one composition factor, a contradiction. $\square$

5.2 Embeddings

Let $0 \leq k \leq n$ and $m = m(p,q)$ be given. Then (k,i) is a middle index for some i if and only if $\frac{1}{2}(n-m+1) < k < \frac{1}{2}(n+m-1)$ and $0 \leq k \leq n$. Let ℓ and r be the minimal and maximal choices for k satisfying this constraint, respectively. It is useful to imagine the middle indices as nodes in an array of $m-1$ rows and $m-1$ or $m-2$ columns, depending on the parity of $n-m$, if $n \geq m-2$. (There will be fewer columns if $n < m-2$ due to the constraint $0 \leq k \leq n$.) Placing the non-zero homologies into this array we obtain the table

$$\mathcal{H}^n: \quad \begin{array}{ccccccc}
H^n_{\ell,1} & H^n_{\ell+1,1} & H^n_{\ell+2,1} & \cdots & 0 & 0 & 0 \\
0 & H^n_{\ell+1,2} & H^n_{\ell+2,2} & \cdots & 0 & 0 & 0 \\
0 & H^n_{\ell+1,3} & H^n_{\ell+2,3} & \cdots & 0 & 0 & 0 \\
0 & 0 & H^n_{\ell+2,4} & \cdots & 0 & 0 & 0 \\
0 & 0 & H^n_{\ell+2,5} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & H^n_{r-2,m-5} & 0 & 0 \\
0 & 0 & 0 & \cdots & H^n_{r-2,m-4} & 0 & 0 \\
0 & 0 & 0 & \cdots & H^n_{r-2,m-3} & H^n_{r-1,m-3} & 0 \\
0 & 0 & 0 & \cdots & H^n_{r-2,m-2} & H^n_{r-1,m-2} & 0 \\
0 & 0 & 0 & \cdots & H^n_{r-2,m-1} & H^n_{r-1,m-1} & H^n_{r,m-1}
\end{array}$$

Figure 2: $\mathcal{H}^n$ -Table

when $n-m$ is even, and a similar table with two non-zero entries in the first and last column, when $n-m$ is odd. (Similarly, remove an equal number of columns on the left and the right of the array if $n < m-2$.) The corner entries $H^n_{r,m-1}$, $H^n_{r-1,m-3}$, $H^n_{r-2,m-5} \cdots$ on the right correspond to maximal middle indices. As discussed in Section 3 the incidence map $\partial\colon M^n_k \to M^n_{k-1}$ induces a map $\partial\colon H^n_{k,i} \to H^n_{k-1,i-1}$. This map is a NW-arrow in the table. The identity map $M^n_k \to M^n_k$ induces a map $\mathrm{inc}\colon H^n_{k,i} \to H^n_{k,i+1}$, this represents a S-arrow in the $\mathcal{H}^n$ -table. Compare to the general situation described in (10).

Lemma 5.3. *(i) Let* $0 \le k \le n$ *and* $1 \le t < i < m$*. Assume that* $2k - t \ge n$*. Then* $\partial^t \colon H^n_{k,i} \to H^n_{k-t,i-t}$ *is an injection.*
(ii) Let $0 \le k \le n$ *and* $1 < i < j < m$*. Assume that* $2k + m - i - j \ge n$*. Then* $\mathrm{inc}^{j-i} \colon H^n_{k,i} \to H^n_{k,j}$ *is an injection.*

Writing $2k - t = k + (k - t)$ the condition in (i) says that the arrow $H^n_{k,i} \to H^n_{k-t,i-t}$ is balanced on or towards the right of the centre of the array. Similarly, the assumption in (ii) is a balancing condition on the diagonal of the array.

Proof: (i) We can assume that $n < 2k + m - i < n + m$ since otherwise $H^n_{k,i} = 0$ by Theorem 4.1, and in this case the assertion is true. Now (k, t) is a middle index if and only if $2k - t < n$. From the assumption and Theorem 4.1 we conclude that $H^n_{k,t} = 0$. For injectivity we show that if $\partial^t(x)$ belongs to $\partial^{m-i+t}(M^n_{k+m-i})$ then x belongs to $\partial^{m-i}(M^n_{k+m-i})$. So suppose that $\partial^t(x) = \partial^{m-i+t}(w)$ for some w in M^n_{k+m-i}. Then $\partial^t\big(x - \partial^{m-i}(w)\big) = 0$ and since $K^n_{k,t} = I^n_{k,t}$ there is some u in M_{k+m-t} so that $x - \partial^{m-i}(w) = \partial^{m-t}(u)$. Hence $x = \partial^{m-i}\big(w + \partial^{i-t}(u)\big)$.

(ii) Again assume that $n < 2k + m - i < n + m$. Now $(k + m - j, m - j + i)$ is a middle index if and only if $2k + m - i - j < n$. From the assumption and Theorem 4.1 we have $H^n_{k+m-j,m-j+i} = 0$. So suppose that $x = \partial^{m-j}(w)$ for some $w \in M^n_{k+m-j}$ and $\partial^i(x) = 0$. Then $\partial^{m-j+i}(w) = 0$ and as $H^n_{k+m-j,m-j+i} = 0$ we have $w = \partial^{j-i}(u)$ for some u. Therefore $x = \partial^{m-j}(\partial^{j-i}(z)) = \partial^{m-i}(u)$, and hence $\mathrm{inc}^{j-i} \colon H^n_{k,i} \to H^n_{k,j}$ is injective. □

Theorem 5.4 (Duality). *Let* $\mathbb{F} = \mathrm{GF}(q)$ *and let* F *be a field of characteristic* $p > 0$ *not dividing* q*. Suppose that* $0 \le k \le n$ *and* $0 < i < m = m(p, q)$*. Then*
(i) $H^n_{k,i} \cong H^n_{n-k,m-i}$*, and*
(ii) $H^n_{k,i} \cong H^n_{k,j}$ *where* $j = 2k - n + m - i$
as FG_n *-modules.*

This is Theorem 2 in the Introduction. It endows the $\mathcal{H}^n$ -array with a $C_2 \times C_2$ symmetry which we will use to complete our analysis.

Example: To illustrate the two dualities consider the case $n = 10$, $m = 5$, $k = 4$ and $i = 2$. Then $j = 1$ and the relevant index pairs are $(4, 2)$, $(6, 3)$, $(4, 1)$ and $(6, 4)$. From the trace formula Corollary 4.3 we can express the $H^n_{k,i}$ as

(a) $H^{10}_{4,2} = M^{10}_4 - M^{10}_2 + M^{10}_9 - M^{10}_7$,
(b) $H^{10}_{6,3} = M^{10}_6 - M^{10}_3 + M^{10}_1 - M^{10}_8$,
(c) $H^{10}_{4,1} = M^{10}_4 - M^{10}_3 + M^{10}_9 - M^{10}_8$,
(d) $H^{10}_{6,4} = M^{10}_6 - M^{10}_2 + M^{10}_1 - M^{10}_7$.

At the level of permutation sets L^n_k is not permutation equivalent to L^n_{n-k}, and so the isomorphism does not hold for permutation sets. However, in the case of cross characteristics we do have $M^n_k \cong M^n_{n-k}$ at the level of permutation modules, see Theorem 14.3 in [4]. From this it follows that the four modules are indeed isomorphic.

Proof of Theorem 5.4: For the index pairs (k, i) and (ℓ, j) we write $(k, i) \hookrightarrow (\ell, j)$ or $(k, i) \leftrightarrow (\ell, j)$ provided that there is an FG_n -monomorphism, or FG_n -isomorphism $H^n_{k,i} \to H^n_{\ell,j}$, respectively. For the first part of the theorem note that $\dim(H^n_{k,i}) = \dim(H^n_{n-k,m-i})$ by Theorem 4.5(iii). In particular, (k, i) is a middle index if and only if $(n - k, m - i)$ is a middle index. Hence it suffices to show that $(k, i) \hookrightarrow (n - k, m - i)$ for all $k \ge \frac{1}{2}n$ and all $0 < i < m$.

When n is even: Here $\mathcal{H}^n$ has a middle column indexed by $\frac{n}{2}$. Consider the middle index (k, i) where $k = \frac{1}{2}n + a$ with $0 \le a < \frac{1}{2}i \le \frac{1}{2}(m - 1)$ since $k < \frac{1}{2}(n + m - 1)$. First suppose that $2i \le m + 2a$. Then $\mathrm{inc}^{m+2a-2i} \colon (k, i) \hookrightarrow (k, m - i + 2a)$ and $\partial^{2a} \colon (k, m - i + 2a) \hookrightarrow (k - 2a, m - i)$ by Lemma 5.3. Hence $(k, i) \hookrightarrow (n - k, m - i)$. Next suppose that $a > 0$ and $2i > m + 2a$. Here

we have $\partial^{2a}\colon (\frac{1}{2}n+a,i) \hookrightarrow (\frac{1}{2}n-a,i-2a)$ by the lemma. Now observe that $(\frac{1}{2}n-a,i-2a) \leftrightarrow (\frac{1}{2}n+a,m-i+2a)$ by the first part of the proof, since $2(m-i+2a) \le m+2a$. Next we have $\partial^{2a}\colon (\frac{1}{2}n+a,m-i+2a) \hookrightarrow (\frac{1}{2}n-a,m-i+2a-2a)$. Hence together $(\frac{1}{2}n+a,i) \hookrightarrow (\frac{1}{2}n-a,m-i)$ which proves the result for n even.

When n is odd: The argument is almost the same. Consider the middle index (k,i) where $k=\frac{1}{2}n+a$ with $a=\frac{1}{2},1+\frac{1}{2},...,< \frac{1}{2}i \le \frac{1}{2}(m-1)$. First suppose that $2i \le m+2a$. Then $\partial^{2a}\colon (k,i)=(\frac{1}{2}n+a,i) \hookrightarrow (\frac{1}{2}n-a,i-2a)$ and $\mathrm{inc}^{m-2i+2a}\colon (\frac{1}{2}n-a,i-2a) \hookrightarrow (\frac{1}{2}n-a,m-i)$ by Lemma 5.3. Hence $(k,i) \hookrightarrow (n-k,m-i)$ in this case. Next suppose that $2i > m+2a$ so that $\partial^{2a}\colon (k,i)=(\frac{1}{2}n+a,i) \hookrightarrow (\frac{1}{2}n-a,i-2a)$. Now $(\frac{1}{2}n-a,i-2a) \leftrightarrow (\frac{1}{2}n+a,m-i+2a)$ by the first part of the proof, as $2(m-i+2a) \le m+2a$. This gives $\partial^{2a}\colon (\frac{1}{2}n+a,m-i+2a) \hookrightarrow (\frac{1}{2}n-a,m-i+2a-2a)$. Hence $(\frac{1}{2}n+a,i) \hookrightarrow (\frac{1}{2}n-a,m-i)$, and this completes the proof of the first part.

To prove the second part note that (k,i) is a middle index if and only if (k,j) with $0<j=2k-n+m-i<m$ is a middle index. Assume therefore that (k,i) is a middle index so that $0<2k-n+m-i<m$ and without loss $i<j$. First suppose $k \ge \frac{1}{2}n$, with $k=\frac{1}{2}n+a$. Then $\mathrm{inc}^{2k-n+m}\colon (k,i) \hookrightarrow (k,2k-n+m-i)=(k,j)$ by Lemma 5.3. Conversely, we have $\partial^{2a}\colon (k,2k-n+m-i) \hookrightarrow (k-2a,2k-2a-n+m-i)$ by the lemma and $(k-2a,2k-2a-n+m-i) \leftrightarrow (n-k+2a,m-2k+2a+n-m+i)$ by the first part of the theorem. But $(n-k+2a,m-2k+2a+n-m+i)=(k,i)$. Hence $(k,j) \hookrightarrow (k,i)$. The case $k \le \frac{1}{2}n$ is very similar. □

By the last theorem all homologies are determined by the modules $H^n_{k,i}$ in the 'triangle' where $2k=n+2a$ with $a \ge 0$ and $2i \le m+2a$. It remains to examine these terms.

Lemma 5.5. *Let (k,i) be a maximal middle index for $n \ge 0$, and write $k=\frac{1}{2}n+a$ with $a \ge 0$. Suppose that j is an integer with $i<j<m$ and $2j \le m+2a$. Then $\beta^n_{k,j}=\beta^n_{k,i}+\beta^n_{k+1,j+1}$.*

Proof: Note, our usual convention applies, we put $\beta^n_{k+1,j+1}=0$ if $(k+1,j+1)$ is not a middle index. The equation holds for $j=m-1$ by Theorem 5.4(ii), and similarly for $a \ge \frac{1}{2}(m-3)$ in which case k indexes the last column in the $\mathcal{H}^n$-array. The statement is also true for $0 \le n \le 1$. The result now follows by induction and Theorem 4.5(i). □

Theorem 5.6. *Let $\mathbb{F}=\mathrm{GF}(q)$ and let F be a field of characteristic $p>0$ not dividing q. Let (k,i) be a maximal middle index for $n \ge 0$, and write $k=\frac{1}{2}n+a$ with $a \ge 0$. Suppose that j is an integer with $i<j<m$ and $2j \le m+2a$. Then $H^n_{k,j} \cong H^n_{k,i} \oplus H^n_{k+1,j+1}$ as FG_n-modules.*

Proof: By Lemma 5.3 we have $\mathrm{inc}^{j-i}\colon (k,i) \hookrightarrow (k,j)$ and $\partial\colon (k+1,j+1) \hookrightarrow (k,j)$. As (k,i) is a maximal middle index it follows from Proposition 5.2 that $H^n_{k,i}=D^{(n-k,k)}$ is irreducible and not a composition factor of $H^n_{k+1,j+1}$. Therefore the sum $\mathrm{inc}^{j-i}(H^n_{k,i})+\partial(H^n_{k+1,j+1}) \subseteq H^n_{k,j}$ is direct and the result follows from Lemma 5.5. □

Theorem 5.7. *Let $\mathbb{F}=\mathrm{GF}(q)$ and let F be a field of characteristic $p>0$ not dividing q. Let (k,j) be a middle index for $n \ge 0$. Assume that $k \ge \frac{1}{2}n$ and $j \le \frac{1}{2}(m-n)+k$. Put $\ell=n-k+j-1$. Then $H^n_{k,j}=\bigoplus_{t=k}^{\ell} D^{(n-t,t)}$.*

When $j>k+1$ then for some terms in $\bigoplus_{t=k}^{\ell} D^{(n-t,t)}$ we have expressions $(n-t,t)$ with $n-t<0$. Here evidently $D^{(n-t,t)}=0$.

Proof: First notice that $\ell \ge k$ since (k,j) is a middle index, with equality if and only if (k,j) is a maximal middle index. Next let k be maximal with $k<\frac{1}{2}(n+m-1)$, that is, k indexes the right-most column of $\mathcal{H}^n$. If $\frac{1}{2}(n+m-1) \le n$ then the constraint $j \le \frac{1}{2}(m-n)+k$ implies that (k,j) is a maximal middle index, and hence $k=\ell$. From Proposition 5.2 it follows that $H^n_{k,j}=D^{(n-k,k)}$ as required. If $k=n$ then $H^n_{k,j}=F=D^{(0,n)}=\bigoplus_{t=k}^{\ell} D^{(n-t,t)}$ having in mind the comment about

expressions $(n-t,t)$ with $n-t<0$. So we may apply induction and suppose that the theorem holds for all values $>k$. Now apply Theorem 5.6 and Propostion 5.2. $\square$

Proof of Theorem 3: The first part is Theorem 4.1 and so we turn to the second part. Let (k,i) be a middle index. (a) If $k \geq \frac{1}{2}n$ and $i \leq \frac{1}{2}(m-n)+k$ set

$$T_{k,i} := \{\, t\colon\ k \leq t \leq n-k+i-1\}.$$

In this case the statement of Theorem 3 is Theorem 5.7. Considering the remaining three possibilities suppose (b) that $k \geq \frac{1}{2}n$ and $i > \frac{1}{2}(m-n)+k$. Here let $i' = 2k-n+m-i$. Then $H^n_{k,i} \cong H^n_{k,i'}$ by Theorem 5.4(ii) and as (k,i') is of type (a) the result follows if we set

$$T_{k,i} = \{\, t\colon\ k \leq t \leq k+m-i-1\}.$$

Similarly, using Theorem 5.4(i) the remaining two cases can be reduced to (a) or (b) if we define

$$\begin{array}{lcll} T_{k,i} & = & \{\, t\colon\ n-k \leq t \leq n-k+i-1\} & \text{if } k < \frac{1}{2}n \text{ and } i \leq \frac{1}{2}(m-n)+k, \text{ and} \\ T_{k,i} & = & \{\, t\colon\ n-k \leq t \leq k+m-i-1\} & \text{if } k < \frac{1}{2}n \text{ and } i > \frac{1}{2}(m-n)+k. \end{array}$$

This completes the proof. In practical terms the result says that the $\mathcal{H}^n$-array is bordered by $D^{(n-k,k)}$ on both ends of column k with the remainder filled in using the simple rule of Theorem 5.7 and its mirror versions. $\square$

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